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The Gravitational Field of a Circulating Light Beam

Ronald L. Mallett[1]

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Exact solutions of the Einstein field equations are found for the exterior and interior gravitational field of an infinitely long circulating cylinder of light. The exterior metric is shown to contain closed timelike lines.

KEY WORDS: general relativity and gravitation; exact solutions; closed timelike lines; lasers.

1. INTRODUCTION

A number of interesting post-Newtonian phenomena are known to occur for rotating distributions of matter in Einstein's general theory of relativity. Inertial frame dragging, for example, is a consequence of the weak gravitational field of a slowly rotating massive sphere.(1) In addition, exact solutions of the Einstein field equations indicate the presence of closed timelike lines for rotating Kerr black holes,(2) van Stockum rotating dust cylinders,(3, 4) and the rotating universe of Godel.(5)

Recently, the author(6) solved the linearized Einstein field equations to obtain the gravitational field produced by the electromagnetic radiation of a unidirectional ring laser. It was shown that a massive spinning neutral particle at the center of the ring laser exhibited inertial frame dragging.

Photonic crystal wave guides with the capability of very sharp light bending (short radius of curvature) have been demonstrated.(7) Directing a laser beam along a helical path to form a solenoid of light in a photonic crystal (see Fig. 1) would enhance the strength of the gravitational field generated by a circulating light beam. A generalized model of the system

[1] Department of Physics, 2152 Hillside Road, University of Connecticut, Storrs, Connecticut 06269; e-mail: RLMallett@aol.com

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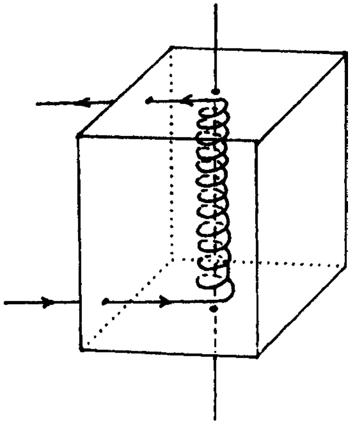

Fig. 1. Solenoidal laser beam through wave guide photonic crystals.

just described would be a long circulating cylinder of light. Such a model is the subject of the present paper.

In this paper exact solutions of the Einstein field equations are found for the exterior and interior gravitational field produced by an infinitely long circulating cylinder of light. The exterior solution is shown to contain closed timelike lines.

2. GRAVITATIONAL FIELD OF CIRCULATING LIGHT

We seek a solution of the Einstein field equations given by ($c = G = 1$)

$$G_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{1}$$

with the energy momentum tensor of axially circulating radiation represented by

$$T_{\mu\nu} = \epsilon \eta_\mu \eta_\nu, \tag{2}$$

where ϵ is the radiation energy density with $x^\mu = (x^0, x^1, x^2, x^3) = (t, \rho, z, \varphi)$ and η_μ a null vector such that

$$\eta_\mu \eta^\mu = 0, \tag{3}$$

with

$$\eta_\mu = (\eta_0, 0, 0, \eta_3). \tag{4}$$

Consider a canonical form of an axisymmetric line element[8, 9]

$$ds^2 = f\,dt^2 - 2w\,dt\,d\varphi - l\,d\varphi^2 - e^{\mu}(d\rho^2 + dz^2), \tag{5}$$

with metric components

$$g_{00} = f, \qquad g_{03} = -w, \qquad g_{33} = -l, \qquad g_{11} = g_{22} = -e^{\mu}, \tag{6}$$

$$g^{00} = \Delta^{-2} l, \qquad g^{03} = -\Delta^{-2} w, \qquad g^{33} = -\Delta^{-2} f, \qquad g^{11} = g^{22} = -e^{-\mu}, \tag{7}$$

where Δ is defined by

$$\Delta^2 \equiv fl + w^2. \tag{8}$$

Since Eq. (1) can be written as

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}, \tag{9}$$

then Eqs. (1)–(3) and Eq. (9) imply

$$R_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{10}$$

$$R = 0. \tag{11}$$

Using Eqs. (6)–(8) to compute the Ricci components R_{00}, R_{03}, and R_{33} for $f = f(\rho)$, $l = l(\rho)$, and $w = w(\rho)$ gives

$$R_{00} = \tfrac{1}{2} e^{-\mu} \Delta [(\Delta^{-1} f_\rho)_\rho + \Delta^{-3} f \Phi], \tag{12}$$

$$R_{03} = -\tfrac{1}{2} e^{-\mu} \Delta [(\Delta^{-1} w_\rho)_\rho + \Delta^{-3} w \Phi], \tag{13}$$

$$R_{33} = -\tfrac{1}{2} e^{-\mu} \Delta [(\Delta^{-1} l_\rho)_\rho + \Delta^{-3} l \Phi], \tag{14}$$

where Φ is defined by

$$\Phi \equiv f_\rho l_\rho + w_\rho^2, \tag{15}$$

with $f_\rho \equiv \partial f / \partial \rho$, $w_\rho = \partial w / \partial \rho, \ldots$. Equations (7) and (11), with $R_{11} = R_{22} = 0$, imply

$$\Delta^{-2}(l R_{00} - 2w R_{03} - f R_{33}) = 0. \tag{16}$$

Equations (12)–(16) yield

$$\Delta_{\rho\rho} = 0. \tag{17}$$

Choosing the solution of Eq. (17) such that $\Delta = \rho$ simplifies the field equations and allows the identification of ρ as the canonical cylindrical coordinate. Equation (8) now has the constraint

$$\Delta^2 = fl + w^2 = \rho^2. \tag{18}$$

Using Eqs. (3), (4), and (6), we have

$$\eta^0 = \xi\eta^3, \tag{19}$$

where

$$\xi = f^{-1}(w + \rho). \tag{20}$$

Now for $T_{\mu\nu} \neq 0$, using Eqs. (2), (6), (10), (20) with $\eta_\mu = g_{\mu\lambda}\eta^\lambda$, we find the following combinations

$$R_{00} + \xi^{-1}R_{03} = 0, \tag{21}$$

$$R_{00} - \xi^{-2}R_{33} = 0, \tag{22}$$

$$R_{03} + \xi^{-1}R_{33} = 0. \tag{23}$$

Substituting Eqs. (12)–(14) in Eqs. (21)–(23), using Eqs. (18) and (20) yields, respectively,

$$\xi(\rho^{-1}f_\rho)_\rho - (\rho^{-1}w_\rho)_\rho + \rho^{-2}\Phi = 0, \tag{24}$$

$$\xi^2(\rho^{-1}f_\rho)_\rho + (\rho^{-1}l_\rho)_\rho + 2\rho^{-2}\xi\Phi = 0, \tag{25}$$

$$\xi(\rho^{-1}w_\rho)_\rho + (\rho^{-1}l_\rho)_\rho + \rho^{-2}\xi\Phi = 0. \tag{26}$$

Equation (20) implies that

$$f\xi = w + \rho. \tag{27}$$

Substituting Eq. (27) in Eq. (24) for ξ constant gives

$$\Phi = f_\rho l_\rho + w_\rho^2 = 1. \tag{28}$$

Using Eq. (28) in Eq. (26) yields

$$(\rho^{-1}W_\rho)_\rho + \rho^{-2} = 0, \tag{29}$$

where

$$W \equiv w + \xi l. \tag{30}$$

A solution of Eq. (29) is given by

$$W = \rho. \tag{31}$$

For points outside the light cylinder $T_{\mu\nu} = 0$, so that Eqs. (10) and (13) together with Eqs. (18) and (28) result in a differential equation for w given by

$$\rho^2 w_{\rho\rho} - \rho w_\rho + w = 0. \tag{32}$$

Equation (32) has as a solution

$$w = \lambda\rho \ln(\rho/\alpha), \tag{33}$$

where α is a positive constant and λ is a constant proportional to the energy density per unit length. Substituting Eq. (33) in Eq. (27), setting the constant $\xi = \alpha$, gives

$$f = \rho/\alpha + \lambda(\rho/\alpha) \ln(\rho/\alpha). \tag{34}$$

The function $l(\rho)$ can be obtained from the constraint Eq. (18) in the form

$$l = (\rho^2 - w^2)/f. \tag{35}$$

Substituting Eqs. (33) and (34) in Eq. (35) gives

$$l = \rho\alpha - \lambda\rho\alpha \ln(\rho/\alpha). \tag{36}$$

It is seen that Eqs. (33) and (36) are consistent with Eqs. (30) and (31) for $\xi = \alpha$. It is also straightforward to show that Eqs. (33), (34), and (36) satisfy $f_\rho l_\rho + w_\rho^2 = 1$ and are thus consistent with Eq. (28).

The Einstein field equations for $R_{11} = 0$ and $R_{22} = 0$ with $\Phi = 1$ give

$$R_{11} = -\tfrac{1}{2}\mu_{\rho\rho} + \tfrac{1}{2}\rho^{-1}\mu_\rho + \tfrac{1}{2}\rho^{-2} = 0, \tag{37}$$

$$R_{22} = -\tfrac{1}{2}\mu_{\rho\rho} - \tfrac{1}{2}\rho^{-1}\mu_\rho = 0. \tag{38}$$

Equations (37) and (38) yield the result

$$e^\mu = (\rho/\alpha)^{-\frac{1}{2}}. \tag{39}$$

Thus, the metric functions f, w, l, and e^μ in Eqs. (33), (34), (36), and (39) give the solution of the Einstein field equations for the exterior of the circulating light cylinder.

We now turn to a discussion of the interior solution. Consider the $\mu = 0$, $\nu = 3$ component of Eq. (10)

$$R_{03} = 8\pi T_{03}, \tag{40}$$

where

$$T_{03} = \epsilon \eta_0 \eta_3. \tag{41}$$

For $\eta^\mu = (1, 0, 0, \xi^{-1})$, using Eqs. (13), (18), (28), and (41), with $\xi = \alpha$ and $\eta_\mu = g_{\mu\lambda}\eta^\lambda$ in Eq. (40), we find

$$(\rho^{-1} w_\rho)_\rho + \rho^{-3} w = 16\pi\epsilon e^\mu \rho/\alpha. \tag{42}$$

We proceed by determining the Green's function which satisfies the equation

$$(\rho^{-1} G_\rho)_\rho + \rho^{-3} G = -\delta(\rho - \rho'). \tag{43}$$

The solution of Eq. (42) is then given by

$$w(\rho) = \frac{16\pi}{\alpha} \int G(\rho, \rho')\, \epsilon e^\mu \rho'\, d\rho', \tag{44}$$

with the Green's function

$$G(\rho, \rho') = \begin{cases} -\rho\rho' \ln \rho', & \rho < \rho', \\ -\rho\rho' \ln \rho, & \rho > \rho'. \end{cases} \tag{45}$$

It can be verified by direct substitution that Eq. (44), integrated from $\rho = 0$ to the boundary ρ_0 of the cylinder, with Eq. (45) is a solution of Eq. (42). Equation (44) with Eq. (45), matched at the boundary with Eq. (33), allows a determination of the relation between the parameters λ and α and ϵ and ρ_0. Using Eqs. (27) and (35) with Eq. (44) yields

$$f = \frac{\rho}{\alpha} + \frac{16\pi}{\alpha^2} \int G(\rho, \rho')\, \epsilon e^\mu \rho'\, d\rho', \tag{46}$$

$$l = \rho\alpha - 16\pi \int G(\rho, \rho')\, \epsilon e^\mu \rho'\, d\rho'. \tag{47}$$

It can be shown that Eqs. (46) and (47) with Eq. (45) satisfy, respectively, the Einstein field equations $R_{00} = 8\pi T_{00}$ and $R_{33} = 8\pi T_{33}$. The interior functions f, w, and l in Eqs. (44), (46), and (47), with $R_{11} = 0$ and $R_{22} = 0$, also satisfy the equation $R = \rho^{-2}(lR_{00} - 2wR_{03} - fR_{33}) = 0$. The function e^μ is again given by Eq. (39).

An interesting consequence of the exterior metric is obtained by considering the line element in Eq. (5) for fixed t, ρ, and z. Equation (5) then reduces to

$$ds^2 = -l\, d\varphi^2. \tag{48}$$

Equation (36) implies that for $\lambda \ln(\rho/\alpha) > 1$ then $l < 0$ so that the curves given by Eq. (48) under these conditions are closed and timelike. Note that under a coordinate transformation $t' = t - \alpha\varphi$ then Eq. (5) with Eqs. (33), (34), and (36) takes the form

$$ds^2 = f\, dt'^2 - 2w'\, dt'\, d\varphi - e^{\mu}(d\rho^2 + dz^2), \tag{49}$$

where $w' = \rho$ with t' a periodic coordinate.

3. CONCLUSION

It has long been known[3,4] that the van Stockum solution for the exterior metric of an infinitely long rotating dust cylinder contains closed timelike lines. The present paper has shown closed timelike curves also occur for an infinitely long circulating cylinder of light. This model also shares some of the same limitations as the van Stockum solution in that the metric is not asymptotically flat. Bonnor,[4] however, has emphasized that certain aspects of an infinitely long rotating dust cylinder may be shared by a long finite one. This may also apply to a long but finite circulating cylinder of light.

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